Exact solutions for N steady fingers in a Hele-Shaw cell

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Exact solutions are reported for N steady fingers moving with a constant speed U in a Hele-Shaw channel in the absence of surface tension. [S1063-651X(98)12610-7]

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The problem of the displacement of a more viscous fluid by a less viscous one in a Hele-Shaw cell has received considerable attention, both theoretically and experimentally, since the pioneering work by Saffman and Taylor [1]. In particular, the situation when surface tension effects are neglected is rather amenable to mathematical investigation and several exact solutions have been found in this case. For example, a large class of solutions for steady bubbles in a Hele-Shaw cell has been reported previously by the present author [2]. Exact time-dependent solutions in the form of the so-called pole dynamics have also been extensively investigated in the literature [3]. Of particular interest among these time-dependent solutions is a class of solutions [4,5] whose asymptotic interface corresponds to N steadily moving fingers. The physical relevance of multifingers solutions has been discussed recently [6] in connection with the role of surface tension in the dynamics of fingering patterns. Although the long-time behavior of the N-finger solution has been extensively studied [7,8], an explicit solution for these N steady fingers is nonetheless warranted. To present such a calculation is the chief aim of this Brief Report.

The problem of N steady fingers moving with a constant speed U in a Hele-Shaw channel in the absence of surface tension is formulated as follows. Let $\mathcal{D}$ denote the region inside the channel occupied by the viscous fluid and $\mathcal{C}_k$ the air-fluid interface corresponding to the kth finger [Fig. 1(a)]. The fluid-velocity field $\mathbf{v}(x,y)$ is given by

$$\mathbf{v}=\nabla\phi, \tag{1}$$

where the velocity potential $\phi(x,y)$ is the solution to the free-boundary problem

$$\nabla^2\phi=0 \quad \text{in} \quad \mathcal{D}, \tag{2a}$$

$$\phi=0, \quad \nabla\phi\cdot\hat{\mathbf{n}}=U\cos\theta \quad \text{on} \quad \mathcal{C}_k, \tag{2b}$$

$$\nabla\phi\cdot\hat{\mathbf{n}}=0 \quad \text{at} \quad y=\pm 1, \tag{2c}$$

$$\phi\approx Vx \quad \text{as} \quad x\rightarrow\infty. \tag{2d}$$

Here $\hat{\mathbf{n}}$ is the outward unit vector normal to the interface $\mathcal{C}_k$, θ is the angle between $\hat{\mathbf{n}}$ and the x axis, and V is the far-field fluid velocity. (Without loss of generality I shall henceforth set $V=1$.) Physically, the velocity potential ϕ is identified (up to a negative constant of proportionality) with the fluid-pressure field.

It is most convenient to analyze the problem in a frame (x',y') moving with fingers. If one introduces the complex variable $z=x'+iy'$, the flow can then be described by the complex potential $W(z)=\phi'(x',y')+i\psi'(x',y')$, where ϕ' is the velocity potential in the moving frame and ψ' is the corresponding stream function. From this point onward, I shall, however, drop the prime notation with the understanding that I will be working solely in the frame where the fingers are stationary. The problem given in Eq. (2) can now be restated as follows: The function $W(z)$ must be analytic in the fluid domain $\mathcal{D}$ and satisfy the boundary conditions

$$W=-Ux+i\psi_k \quad \text{on} \quad \mathcal{C}_k, \quad k=1,2,\ldots,N, \tag{3}$$

$$\text{Im}\,W=\pm(1-U) \quad \text{at} \quad y=\pm 1, \tag{4}$$

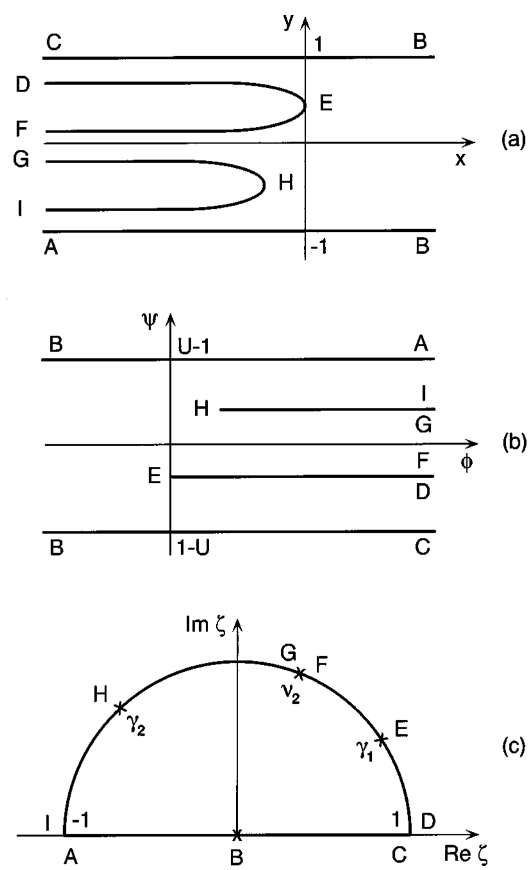

FIG. 1. Flow geometry: (a) the z plane, (b) the W plane, and (c) the ζ plane.

where the ψ_k's are real-valued constants and Im W indicates the imaginary part of W. (For a more detailed discussion of these boundary conditions see Ref. [2].) We can thus view $W(z)$ as a conformal mapping from the fluid domain $\mathcal{D}$ in the z plane to a region in the W plane that consists of an infinite strip of width $2(U-1)$ with N horizontal slits [Fig. 1(b)].

Next consider the conformal mapping $z=f(\zeta)$ from the interior of the unit semicircle in the complex ζ plane [Fig. 1(c)] to the fluid domain $\mathcal{D}$ in the z plane. For conciseness, but admittedly in an abuse of notation, I shall write $W(\zeta)$ for $W(f(\zeta))$. Thus $W(\zeta)$ maps conformally the interior of the unit semicircle in the ζ plane onto the flow domain in the W plane. Such a mapping can be easily constructed and one finds that $W(\zeta)$ is given by

$$W_\zeta=\frac{2(U-1)}{\pi}\,\frac{\prod_{k=1}^{N}(\zeta-e^{i\gamma_k})(\zeta-e^{-i\gamma_k})}{\zeta(1-\zeta^2)\prod_{k=2}^{N}(\zeta-e^{i\nu_k})(\zeta-e^{-i\nu_k})},\quad(5)$$

where the ζ subscript denotes derivative with respect to ζ. Here γ_k and ν_k are real-valued parameters taking values in the range $0=\nu_1<\gamma_1<\nu_2<\gamma_2<\nu_3<\cdots<\gamma_N<\nu_{N+1}=\pi$, where for convenience I have also introduced the parameters $\nu_1=0$ and $\nu_{N+1}=\pi$. As indicated in Fig. 1, the points $\zeta=\exp(i\gamma_k)$ and $\zeta=\exp(i\nu_k)$ are mapped, respectively, on the tips and end points of the fingers.

The mapping function $f(\zeta)$ can now be obtained by first noting that on $|\zeta|=1$ we have $x=\frac{1}{2}(z+\overline{z})=\frac{1}{2}[f(\zeta)+f(1/\zeta)]$. Inserting this identity into the condition (3) yields the equality

$$f(\zeta)+f(1/\zeta)=-\frac{2}{U}W(\zeta),\quad(6)$$

which is valid on $|\zeta|=1$ and, by analytic continuation, inside the unit semicircle as well. Using the fact that $W(1/\zeta)=W(\zeta)$, one can then easily verify that the solution to Eq. (6) is given by

$$f(\zeta)=-\frac{1}{U}\left[\frac{2}{\pi}\ln\zeta+W(\zeta)\right].\quad(7)$$

Integrating Eq. (5) to yield $W(\zeta)$ and inserting this into Eq. (7), one finally obtains

$$f(\zeta)=-\frac{2}{\pi}\ln\zeta+\frac{2}{\pi}(1-U^{-1})\Bigg[a_1\ln(1-\zeta)+a_{N+1}\ln(1+\zeta)+\sum_{k=2}^{N}a_k\ln(\zeta-e^{i\nu_k})(\zeta-e^{-i\nu_k})\Bigg].\quad(8)$$

Here the coefficients a_k, in terms of the original parameters γ_k and ν_k, are given by

$$a_k=\frac{\prod_{l=1}^{N}(\cos\nu_k-\cos\gamma_l)}{\prod_{l=2,\,l\neq k}^{N}(\cos\nu_k-\cos\nu_l)}\quad(9)$$

for $k=1,\ldots,N+1$. The interface shape for each of the N fingers is then obtained from the parametric equations

$$x_k(\theta)+iy_k(\theta)=f(e^{i\theta}),\quad \nu_k<\theta<\nu_{k+1},\quad k=1,\ldots,N.\quad(10)$$

Equations (8)–(10) thus give a $2N$-parameter (corresponding to N parameters γ_k, $N-1$ parameters ν_k, and the speed U) family of solutions for N steady fingers in a Hele-Shaw channel in the absence of surface tension. Note that for a given set of parameters $\{\gamma_k,\ \nu_k\}$ solutions for arbitrary velocity $U>1$ are possible. (This is the characteristic degeneracy of Hele-Shaw flows without surface tension [1].) Here, however, we need to concern ourselves only with the solutions for $U=2$. Solutions for any value $U>1$ can be obtained from a mere rescaling of the $U=2$ solutions, as shown in Ref. [2]. More precisely, if $(\hat{x}_k(\theta),\hat{y}_k(\theta))$ denotes a particular solution with $U=2$, then there will exist a corresponding solution for an arbitrary $U>1$ whose parametric equations are [2]

$$x_k(\theta)=(1+\mu)\hat{x}_k(\theta),\quad(11)$$

$$y_k(\theta)=(1-\mu)\hat{y}_k(\theta),\quad(12)$$

where $\mu=1-2U^{-1}$.

Setting $U=2$ in Eq. (8) and evaluating $f(e^{i\theta})$, one obtains after some simplification that the interface corresponding to the kth finger has the equation

$$x_k=\frac{a_1}{\pi}\ln\sin\frac{\pi}{2}(y_k^0-y)+\frac{a_{N+1}}{\pi}\ln\cos\frac{\pi}{2}(y_k^0-y)+\frac{1}{\pi}\sum_{l=2}^{N}a_l\ln[\cos\pi(y_k^0-y)-\cos\nu_l],\quad(13)$$

where

$$y_k^0=1-\frac{a_1}{2}-\sum_{l=2}^{k}|a_l|,\quad(14)$$

and the variable y (over the kth finger) is restricted to the interval

$$\frac{\nu_k}{\pi}<y_k^0-y<\frac{\nu_{k+1}}{\pi},\quad k=1,\ldots,N.\quad(15)$$

In Eq. (13) an additive constant has been removed by appropriately redefining the origin of the x axis.

A geometrical interpretation of the solution parameters is now straightforward. From Eq. (15) it immediately follows that the relative width λ_k of the kth finger with respect to the channel width is given by

$$\lambda_k = \frac{1}{2\pi}(\nu_{k+1} - \nu_k), \quad k = 1,\ldots,N. \tag{16}$$

From Eq. (8) one also readily sees that the coefficients a_k fix the separation (''gaps'') between adjacent fingers. More specifically, the relative gap δ_k (with respect to the channel width) between the $(k-1)$th and kth finger is simply

$$\delta_k = \frac{|a_k|}{2}, \quad k = 2,\ldots,N. \tag{17}$$

Similarly, $\delta_1 = \frac{1}{4}a_1$ is the relative separation between the first finger (from top down) and the upper sidewall, whereas $\delta_{N+1} = \frac{1}{4}a_{N+1}$ is the corresponding gap between the Nth finger and the lower sidewall. Let us now denote by $\Lambda = \sum_{k=1}^{N}\lambda_k$ the total relative width of all the fingers combined and by $\Delta = \sum_{k=1}^{N}\delta_k$ the total portion occupied by the fluid (on the far left side of the channel), so that $\Lambda + \Delta = 1$. For a solution with $U = 2$ we must have $\Lambda = \Delta = 1/2$ which in turn implies that the parameters a_k must satisfy the additional condition

$$a_1 + a_{N+1} + 2\sum_{k=2}^{N}|a_k| = 2. \tag{18}$$

Thus all possible solutions can be conveniently constructed by prescribing the widths of the N fingers (which determine the parameters ν_k) together with the gaps between adjacent fingers (which fix the coefficients a_k). Shown in Fig. 2 is a solution for $N = 3$ with $\lambda_1 = 0.24$, $\lambda_2 = 0.11$, and $\lambda_3 = 0.15$.

Finally, I note that the solution for an asymmetrical finger originally obtained by Taylor and Saffman [9] can be recovered from our generic solution (13) by simply setting $N = 1$. This yields

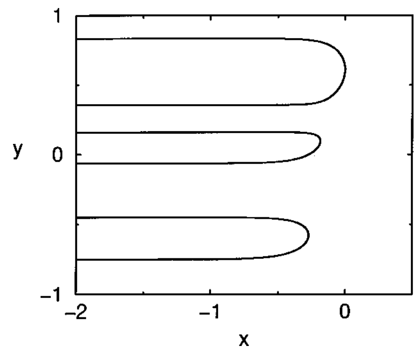

FIG. 2. Solution for $N = 3$ with $\nu_2 = 1.5$, $\nu_3 = 2.2$, $a_1 = 0.3$, $a_2 = 0.2$, and $a_3 = 0.4$.

$$x = \frac{a_1}{\pi}\ln\cos\frac{\pi}{2}\left(\frac{a_2}{2} - y\right) + \frac{a_2}{\pi}\ln\sin\frac{\pi}{2}\left(\frac{a_2}{2} - y\right), \tag{19}$$

where the parameters a_1 and a_2 must satisfy the condition $a_1 + a_2 = 2$. Introducing the asymmetry parameter $y_0 = \frac{1}{4}(a_2 - a_1)$, Eq. (19) can then be rewritten as

$$x = \frac{1}{\pi}\ln\cos\pi(y - y_0) + \frac{2y_0}{\pi}\ln\tan\left[\frac{\pi}{4} + \frac{\pi}{2}(y - y_0)\right], \tag{20}$$

where an additive constant has been removed so as to conform with the original choice of coordinates by Taylor and Saffman [9]. Equation (20) thus reproduces the original Taylor-Saffman finger [see Eq. (21) in Ref. [9]].

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